

About a low complexity class of Cellular Automata*

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Résumé

Extending to all probability measures the notion of μ -equicontinuous cellular automata introduced for Bernoulli measures by Gilman, we show that the entropy is null if μ is an invariant measure and that the sequence of image measures of a shift ergodic measure by iterations of such automata converges in Cesàro mean to an invariant measure μ_c . Moreover this cellular automaton is still μ_c -equicontinuous and the set of periodic points is dense in the topological support of the measure μ_c . The last property is also true when μ is invariant and shift ergodic.

1 Introduction, definitions

Let A be a finite set. We denote by $A^{\mathbb{Z}}$, the set of bi-infinite sequences $x = (x_i)_{i \in \mathbb{Z}}$ where $x_i \in A$. We endow $A^{\mathbb{Z}}$ with the product topology of the discrete topologies on A . A point $x \in A^{\mathbb{Z}}$ is called a configuration. The shift $\sigma: A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$ is defined by $\sigma(x) = (x_{i+1})_{i \in \mathbb{Z}}$. A cellular automaton (CA) is a continuous self-map F on $A^{\mathbb{Z}}$ commuting with the shift. The Curtis-Hedlund-Lyndon theorem states that for every cellular automaton F there exist an integer r and a block map f from A^{2r+1} to A such that $F(x)_i = f(x_{i-r}, \dots, x_i, \dots, x_{i+r})$. The integer r is called the radius of the cellular automaton. For integers i, j with $i \leq j$ we denote by $x(i, j)$ the word $x_i \dots x_j$ and by $x(i, \infty)$ the infinite sequence $(v_n)_{n \in \mathbb{N}}$ such that for all $n \in \mathbb{N}$ one has $v_n = x_{i+n}$. For any integer $n \geq 0$ and point $x \in A^{\mathbb{Z}}$, we denote by $B_n(x)$ the set of points y such that for all $i \in \mathbb{N}$, one has $F^i(x)(-n, n) = F^i(y)(-n, n)$ and by $C_n(x)$ the set of points y such that $y_j = x_j$ with $-n \leq j \leq n$. A point $x \in A^{\mathbb{Z}}$ is called an equicontinuous point if for all positive integer n there exists another positive integer m such that $B_n(x) \subset C_m(x)$. A point x is μ -equicontinuous if for all $m \in \mathbb{N}$ one has

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$\lim_{n \rightarrow \infty} \frac{\mu(C_n(x) \cap B_m(x))}{\mu(C_n(x))} = 1$. In this paper, we call μ -equicontinuous CA any cellular automaton with a set of full measure of μ -equicontinuous points. Clearly an equicontinuous point which belongs to $S(\mu) = \{x \in A^{\mathbb{Z}} | \mu(C_n(x)) > 0 | \forall n \in \mathbb{N}\}$, (the topological support of μ) is also a μ -equicontinuous point. When μ is a shift ergodic measure, the existence of μ -equicontinuous points implies that the cellular automaton is μ -equicontinuous (see [2]).

These definitions were motivated by the work of Wolfram (see [6]) that have proposed a first empirical classification based on computer simulations. In [2] Gilman introduces a formal and measurable classification by dividing the set of CA in three parts (CA with equicontinuous points, CA without equicontinuous points but with μ -equicontinuous points, μ -expansive CA). The Gilman's classes are defined thanks to a Bernoulli measure not necessarily invariant and corresponds to the Wolfram's simulations based on random entry. Here we study some properties of the μ -equicontinuous class that allows to construct easily invariant measures (see Theorem 2.1) and we try to describe what kind of dynamic characterizes μ -equicontinuous CA when μ is an invariant measure. Finally, remark that the comparison between equicontinuity (see some properties of this class in [1] and [4]) and μ -equicontinuity take more sense when we study the restriction of the automaton to $S(\mu)$ (see Section 4 for comments and examples).

2 Statement of the results

2.1 Gilman's Results

Proposition 2.1 [3] *If $\exists x$ and $m \neq 0$ such that $B_n(x) \cap \sigma^{-m}B_n(x) \neq \emptyset$ with $n \geq r$ (the radius of the automaton F) then the common sequence $(F^i(y)(-n, n))_{i \in \mathbb{N}}$ of all points $y \in B_n(x)$ is ultimately periodic.*

In [3] Gilman states the following result for any Bernoulli measure μ . The proof uses only the shift ergodicity of these measures and can be extended to any shift ergodic measure.

Proposition 2.2 [3] *Let μ be a shift ergodic measure. If a cellular automaton F has a μ -equicontinuous point, then for all $\epsilon > 0$ there exists a F -invariant closed set Y such that $\mu(Y) > 1 - \epsilon$ and the restriction of F to Y is equicontinuous.*

2.2 New Results

Proposition 2.3 *The measure entropy $h_\mu(F)$ of a μ -equicontinuous and μ -invariant cellular automaton F (with μ not necessarily shift invariant) is equal to zero.*

Proposition 2.4 *If a cellular automaton F has some μ -equicontinuous points where μ is a F -invariant and shift ergodic measure then the set of F -periodic points is dense in the topological support of μ .*

Theorem 2.1 *Let μ be a shift-ergodic measure. If a cellular automaton F has some μ -equicontinuous points then the sequence $(\mu_n)_{n \in \mathbb{N}} = (\frac{1}{n} \sum_{i=0}^{n-1} \mu \circ F^{-i})_{n \in \mathbb{N}}$ converges vaguely to an invariant measure μ_c .*

Theorem 2.2 *If μ is a shift ergodic measure and F a μ -equicontinuous cellular automaton then F is also a μ_c -equicontinuous cellular automaton.*

Corollary 2.1 *If $\mu_c = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu \circ F^{-i}$ where μ is a shift ergodic measure and F is a cellular automaton with μ -equicontinuous points then the set of F -periodic points is dense in $S(\mu_c)$.*

3 Proofs (Sketches)

3.1 Proof of Proposition 2.3

Denote by $(\alpha_p)_{p \in \mathbb{N}}$ the partition of $A^{\mathbb{Z}}$ by the $2p+1$ central coordinates and remark that $h_\mu(F) = \lim_{p \rightarrow \infty} h_\mu(F, \alpha_p)$ where $h_\mu(F, \alpha_p)$ denote the measurable entropy with respect to the partition α_p . Using the Shannon-McMillan-Breiman Theorem, we can show that $\forall p \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $h_\mu(F, \alpha_p) \leq \int \lim_{n \rightarrow \infty} \frac{-\log \mu(B_m(x))}{n} d\mu(x) = 0$.

3.2 Proof of Theorem 2.1

It is sufficient to show that for all $x \in S(\mu)$ and $m \in \mathbb{N}$ the sequence $(\mu_n(C_m(x)))_{n \in \mathbb{N}}$ converges. From Proposition 2.2 there exists a set Y_ϵ of measure greater than $1 - \epsilon$ such that for all points $y \in Y_\epsilon$ and positive integer k the sequences $(F^n(y)(-k, k))_{n \in \mathbb{N}}$ are eventually periodic with preperiod $pp_\epsilon(k)$ and period $p_\epsilon(k)$. We get that $\mu_n(C_m(x) \cap Y_\epsilon) = \frac{1}{n} \sum_{i=0}^{pp_\epsilon(k)-1} \mu(F^{-i}(C_m(x)) \cap Y_\epsilon) + \frac{1}{n} \sum_{i=pp_\epsilon(k)}^{n-1} \mu(F^{-i}(C_m(x)) \cap Y_\epsilon)$ for all $x \in A^{\mathbb{Z}}$ and integer $k \geq m$. Remark that the first term tends to 0 and the periodicity of the second one implies that $\lim_{n \rightarrow \infty} \mu_n(C_m(x) \cap Y_\epsilon) = \frac{1}{p_\epsilon(k)} \sum_{i=0}^{p_\epsilon(k)-1} \mu(F^{-(i+pp_\epsilon(k))}(C_m(x) \cap Y_\epsilon))$. Moreover we have $\lim_{\epsilon \rightarrow 0} \mu_n(C_m(x) \cap Y_\epsilon) = \mu_n(C_m(x))$. Since for all x and $m \in \mathbb{N}$ one has $|\mu_n(C_m(x) \cap Y_\epsilon) - \mu_n(C_m(x))| \leq \frac{n\epsilon}{n} = \epsilon$ the convergence is uniform with respect to ϵ . It follows that we can reverse the limits and obtain that

$$\begin{aligned} \mu_c &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu \circ F^{-i}(C_m(x)) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \lim_{\epsilon \rightarrow 0} \mu \circ F^{-i}(C_m(x) \cap Y_\epsilon) \\ &= \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu \circ F^{-i}(C_m(x) \cap Y_\epsilon) \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{p_\epsilon(k)} \sum_{i=0}^{p_\epsilon(k)-1} \mu(F^{-(i+pp_\epsilon(k))}(C_m(x) \cap Y_\epsilon)) = \mu_c(C_m(x)). \end{aligned}$$

The invariance of converging subsequences of $(\mu_n)_{n \in \mathbb{N}}$ is a classical result. \square

3.3 Proof of Proposition 2.4

Since μ is a shift ergodic measure and there exist a μ -equicontinuous points x , for all $m \in \mathbb{N}$ and $z \in S(\mu)$ there exist $(i, j) \in \mathbb{N}^2$ such that $\mu(C_p(z) \cap \sigma^{-(i+p)} B_r(x) \cap \sigma^{j+p} B_r(x) =: S) > 0$ (r is the radius of the CA). From the Poincaré recurrence theorem, for all $z \in S(\mu)$, there exists $m \in \mathbb{N}$ and $y \in S$ such that $F^m(y)(-r-p-i, j+p-r-1) = y(-r-p-i, j+p-r-1)$. From the Proof of Proposition 2.1 (see [3]), the shift periodic point $\bar{w} = \dots www \dots$ such that $\bar{w}(-r-p-i, j+p-r-1) = w = y(-r-p-i, j+p-r-1)$ belongs to S and since the F orbit of each $y' \in S \cap \{y'' \in A^{\mathbb{Z}} \mid y_l'' = y_l \mid (-r-p-i \leq l \leq j+p-r-1)\}$ share the same central coordinates, it follows that $F^m(\bar{w})(-r-p-i, j+p-r-1) = w = \bar{w}(-r-p-i, j+p-r-1)$ which implies that $F^m(\bar{w}) = \bar{w}$ and permit to conclude.

3.4 Proof of Theorem 2.2 and Corollary 2.1

Let x be a μ -equicontinuous point. For all $m \in \mathbb{N}$, define $Y_m := \bigcup_{i,j \in \mathbb{N}^2} (\sigma^{-i-m} B_r(x) \cap \sigma^{j+m} B_r(x))$ (r is the radius of F) and $\Omega_m = \lim_{n \rightarrow \infty} \bigcap_{j=0}^n \bigcup_{i=j}^{\infty} F^i(Y_m)$ (the omega-limit set of Y_m under F). Since μ is a shift ergodic measure and $\mu(B_r(x)) > 0$, for all $m \in \mathbb{N}$, we get that $\mu(Y_m) = 1$ and consequently $\mu_c(\Omega_m) = 1$. Let $\Lambda(F)$ be the omega-limit set of $A^{\mathbb{Z}}$. Using the eventual periodicity of $(F^n(x)(-r, r))_{n \in \mathbb{N}}$ (see Proposition 2.1), it can be proved that the omega-limit set of $B_r(x)$ is a finite union of sets $B_r(z_l) \cap \Lambda(F)$ ($0 \leq l \leq p-1$). This implies that $\Omega_m = \bigcup_{z \in [z_0 \dots z_{p-1}]} \bigcup_{i,j \in \mathbb{N}^2} (\sigma^{-i-m} B_r(z) \cap \sigma^{j+m} B_r(z)) \cap \Lambda(F)$ and it follows that for all $z \in S(\mu_c)$ and $k \in \mathbb{N}$, the inequality $\mu_c(C_k(z) \cap \Omega_k) > 0$ implies that there always exist a point z' and integers $i, j \geq m$ such that $\mu_c(C_p(z) \cap \sigma^{-(i+p)} B_r(z') \cap \sigma^{j+p} B_r(z')) > 0$. Using final arguments of the proof of Proposition 2.4, the last inequality is sufficient to show Corollary 2.1. For any measurable set E , define $E^{\mu_c} = \{y \in E \mid \lim_{n \rightarrow \infty} \frac{\mu_c(C_n(y) \cap E)}{\mu_c(C_n(y))} = 1\}$. For all $m \in \mathbb{N}$, define $\Omega'_m := \bigcup_{z \in [z_0 \dots z_{p-1}]} \bigcup_{i,j \in \mathbb{N}^2} (\sigma^{-i-m} B_r(z) \cap \sigma^{j+m} B_r(z))^{\mu_c} \cap \Lambda(F)$ and denote by Ω the set $\bigcap_{m \in \mathbb{N}} \Omega'_m$. Since for all measurable set E , one has $\mu_c(E^{\mu_c}) = \mu_c(E)$, for all $m \in \mathbb{N}$, we get that $\mu_c(\Omega'_m) = 1$ and consequently $\mu_c(\Omega) = 1$. Since for all $y \in \Omega$ and $k \in \mathbb{N}$ there exist integers $i, j \geq k$ and a point z' such that $y \in \sigma^{-i}(B_r(z') \cap \sigma^j B_r(z'))^{\mu_c}$, we obtain that $y \in B_m^{\mu_c}(y)$ which finish the proof.

4 Example of μ -equicontinuous CA without equicontinuous points

In [2] Gilman gives an example of a μ -equicontinuous CA F_s that has no equicontinuous points. The automaton F_s act on $\{0, 1, 2\}^{\mathbb{Z}}$ and is defined thank to the following block map of radius 1 :

$$\left| \begin{array}{c|c|c|c|c|c|c|c|c} *00 & *01 & *02 & *10 & *11 & *12 & *20 & *21 & *22 \\ \hline 0 & 1 & 0 & 0 & 1 & 0 & 2 & 0 & 2 \end{array} \right|$$

The letter $*$ stands for any letter in $\{0, 1, 2\}$. Considering 0 as a background element, the 2's move straight down, 1's move to the left and 1 and 2 collide annihilate each other. In this case the measure μ is a Bernoulli measure on $\{0, 1, 2\}^{\mathbb{Z}}$ and the existence of μ -equicontinuous points depends on the parameters $p(0), p(1), p(2)$ of this measure. In [2] it is shown that if $p(2) > p(1)$ then the probability that a 2 is never annihilated is positive and this implies that there exist μ -equicontinuous points. Since the existence or non existence of a sufficient number of 1 in the right side can always modify the central coordinates one has $C_m(x) \not\subset B_n(x)$ for all $n, m \in \mathbb{N}$ which implies that there is no equicontinuous points.

Remark that using Theorem 2.1 and 2.2 the automaton F_s is μ_c -equicontinuous if $p(2) > p(1)$ but the restriction of F_s to $S(\mu_c)$ always has equicontinuous points ($S(\mu_c) = \{0, 2\}^{\mathbb{N}}$ and $F : S(\mu_c) \rightarrow S(\mu_c)$ is the identity). In [5], we describe a more complex CA \mathcal{F} such that $\mathcal{F} : S(\mu_c) \rightarrow S(\mu_c)$ is μ_c -equicontinuous, without equicontinuous points and the invariant measure μ_c is construct thanks to Theorem 2.1.

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